# Alcuin Numbers

## River Crossing Problems

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# The Alcuin Number Problem

**Goal**

For any given configuration of animals, what is the smallest boat size required to transport them across the river?

# Graph Representation

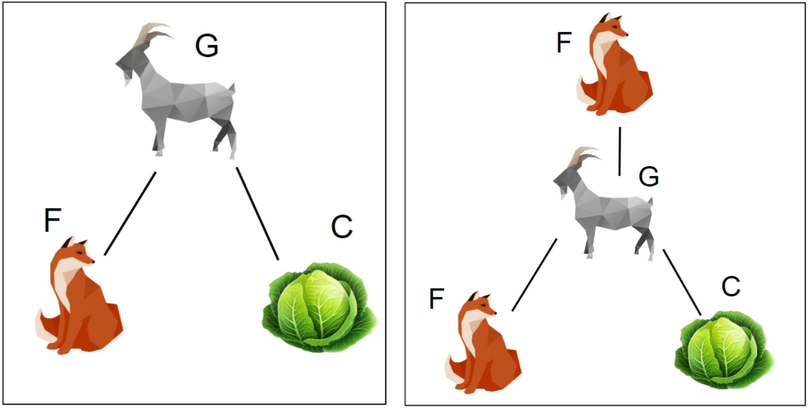


Figure: The Fox-Goat-Cabbage Problem

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# Vertex Covers

## Remark

**The smallest vertex cover $\tau(G)$:** How many vertices (through their adjacent edges) account for all the edges in our graph $G$?

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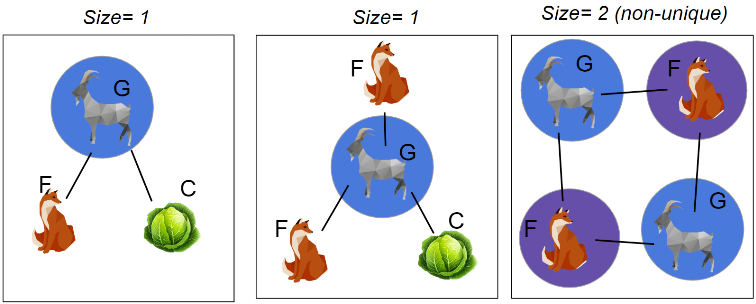



Figure: Vertex Covers

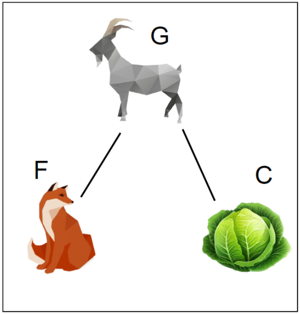
# The Lower Bound on Boat Size

**Remark**

The smallest possible boat is *equal to the size of the vertex cover.*

$$\mathrm{ALCUIN}(G) \geq \tau(G)$$

In fact, you must start by picking up the vertex cover!

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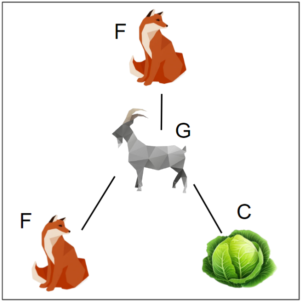
# The Upper Bound on Boat Size

**Proposition**

The largest necessary boat is *equal to the size of the vertex cover + 1.*

$$\mathrm{ALCUIN}(G) \leq \tau(G) + 1$$



# The Alcuin Number Problem [1]

**Remark**

Our problem is now reduced to: $\tau(G) \leq \textbf{ALCUIN}(G) \leq \tau(G) + 1$.

Boats with $\mathrm{ALCUIN}(G) = \tau(G)$ are *Small Boats.*
Boats with $\mathrm{ALCUIN}(G) = \tau(G) + 1$ are *Large Boats.*

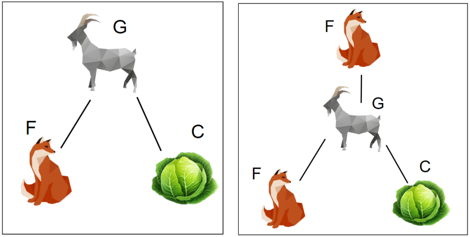



Figure: The Fox-Goat-Cabbage Problem

# Updated Goal

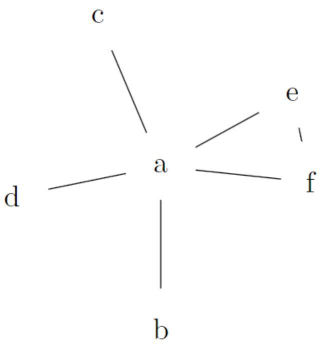
**Goal**

Which graphs are small boat ($|\tau(G)|$) and which graphs are large boat ($|\tau(G)| + 1$)?

# Non-Unique Vertex Covers [1]

## Lemma

If a graph $G$ has more than one minimal vertex cover $\tau(G)$, it is always small boat.

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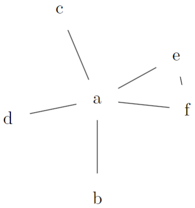
# Non-Unique Vertex Covers- Example Schedule




| Sr No. | Left Bank | Boat | Right Bank |
| --- | --- | --- | --- |
| (1) | $a, b, c, d, e, f$ | $\emptyset$ | $\emptyset$ |
| (2) | $b, c, d, f$ | $a, e$ | $\emptyset$ |
| (3) | $b, c, d, f$ | $a$ | $e$ |
| | $\vdots$ | $\vdots$ | $\vdots$ |
| (4) | $f$ | $a$ | $e, b, c, d$ |
| (5) | $\emptyset$ | $a, f$ | $e, b, c, g$ |
| (6) | $\emptyset$ | $\emptyset$ | $a, f, e, b, c, g$ |

# Updated Goal

**Goal**

Which graphs with a unique vertex cover are small boat or large boat?

# Background- Cliques and Anti-Cliques [2]

## Definition

For $n \in \mathbb{N}$, the **clique** $K_n$ is the *complete* graph $G$ on $n$ vertices. This means that every vertex is connected to every other vertex in $G$.

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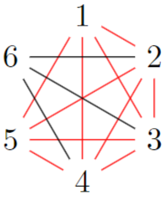


# Representing a graph $G$

Every graph can be separated into vertices of two kinds.

Vertices $v \in \tau(G)$

Vertices $v \in \overline{\tau(G)}$

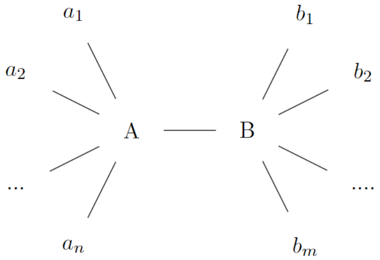



Figure: Graph Representation

# Universal Vertices

## Definition

**A universal vertex** $u \in G$ is a vertex $u \in \overline{\tau(G)}$ such that $u$ is adjacent to every vertex in $\tau(G)$.

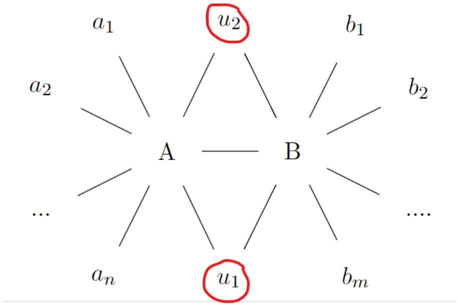



Figure: Universal Vertices

# The Maximum No. Of Universal Vertices [1]

## Lemma

Let $G = (V, E)$ has $\overline{\tau(G)}$ of size $m$ that consists of only universal vertices $u_1, \ldots, u_m$.

1. If $m \leq 2|\mathbb{A}|$, then $G$ is small boat.
2. If $m > 2|\mathbb{A}|$, then $G$ is large boat.

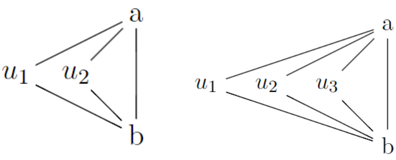
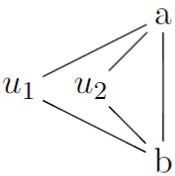


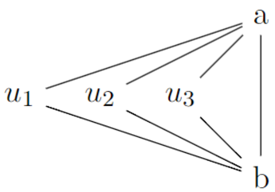

Figure: Non-Unique Vertex Covers are Small Boat

# The Maximum No. Of Universal Vertices- Example




| | | | |
|---|---|---|---|
| (1) | $a, b, u_1, u_2$ | $\emptyset$ | $\emptyset$ |
| (2) | $u_1, u_2$ | $a, b$ | $\emptyset$ |
| (3) | $u_1, u_2$ | $b$ | $a$ |
| (4) | $u_2$ | $u_1, b$ | $a$ |
| (5) | $u_2$ | $a, b$ | $u_1$ |
| (6) | $a$ | $u_2, b$ | $u_1$ |
| (7) | $a$ | $b$ | $u_1, u_2$ |
| (8) | $\emptyset$ | $a, b$ | $u_1, u_2$ |
| (9) | $\emptyset$ | $\emptyset$ | $a, b, u_1, u_2$ |

# The Maximum No. Of Universal Vertices- Example Cntd.




| | | | |
|---|---|---|---|
| (1) | $a, b, u_1, u_2, u_3$ | $\emptyset$ | $\emptyset$ |
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| (3) | $u_1, u_2, u_3$ | $b$ | $a$ |
| (4) | $u_2, u_3$ | $u_1, b$ | $a$ |
| (5) | $u_2, u_3$ | $a, b$ | $u_1$ |

Figure: The Maximal Number of Universal Vertices

# Induced Subgraphs [2]

**Definition**

An **Induced Subgraph** $G' = (V', E')$ of a graph $G = (V, E)$ is a subgraph where $V' \subset V$ and an edge $e'$ exists between two vertices in $G'$ if it also exists in $G$.

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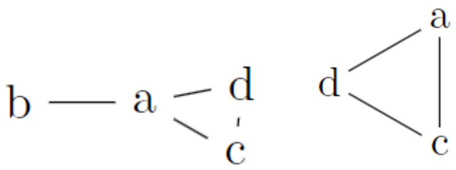


Figure: Induced Subgraphs

# $\tau-$Induced Subgraphs

## Definition

A $\tau-$**Induced Subgraph** $G' = (V', E')$ is an induced subgraph of $G = (V, E)$ where:

1. $\tau(G') \subseteq \tau(G)$
2. If $v \in \overline{\tau(G)}$ is adjacent to $\tau(G')$, then $v \in \overline{\tau(G')}$

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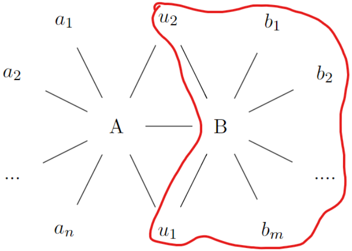


Figure: $\tau$-Induced Subgraphs

# Small Boat Subgraphs (SBSs)

## Definition

Let $G'$ be a graph. $G'$ is an **A Small-Boat-Subgraph (SBS)** if the following condition holds.
**If $G'$ is a $\tau$-Induced Subgraph of another graph $G$, then $G$ is a small boat graph.**

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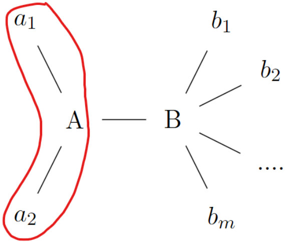



Figure: Small Boat Subgraphs

# Updated Goal

## Goal

**Every small boat graph contains an SBS (which may be itself)**.
Therefore, we need to **categorize all the Small Boat Subgraphs** to determine which graphs are small boat.

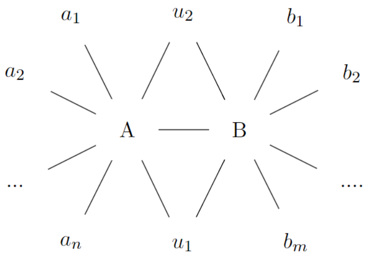
# Maximal Sets of Alcuin graphs

**Definition**

**A maximal set** $[G]$ is a set of small boat graphs such that:

- If we add a vertex $v$ without changing $\tau(G)$
- Or an edge $e$ between two vertices in $\tau(G)$

The resulting graph is **still in the set** OR **large boat**.



# The case of $|\tau(G)| = 1$

There is only one SBS with $|\tau(G)| = 1$, as every vertex is universal.

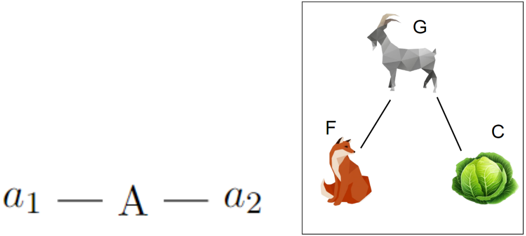



Figure: $|\tau(G)| = 1$

# The case of $|\tau(G)| = 2$

Using Lemma 2- about the maximum number of universal vertices we have:

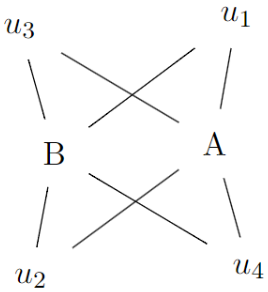



Figure: $|\tau(G)| = 2$

# The case of $|\tau(G)| = 2$

Removing one universal vertex $u_4$, we have:

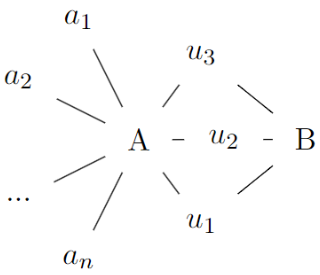



Figure: $|\tau(G)| = 2$

# The case of $|\tau(G)| = 2$

Removing another universal vertex $u_3$, we have:

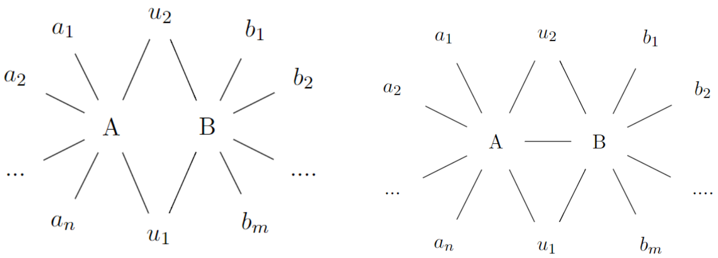



Figure: $|\tau(G)| = 2$

# The complete list of of $|\tau(G)| = 1$ and $|\tau(G)| = 2$

To summarize:

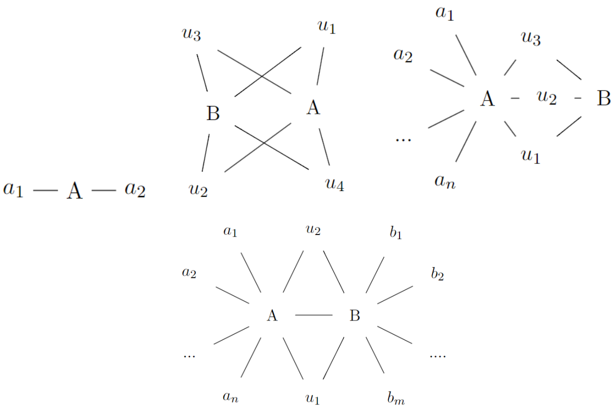



Figure: $|\tau(G)| = 1$ and $|\tau(G)| = 2$

# The complete list of of $|\tau(G)| = 3$